Competitive Outcomes, Endogenous Firm Formation and the Aspiration Core

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Abstract

The paper shows that the aspiration core of any TU-game coincides with the set of competitive wages of two different types of production economy. If agents can divide their time among various coalitions, the set of firms that are active in the market is endogenously determined in equilibrium and it coincides with the generating collection of the corresponding aspiration core allocation. Assuming indivisible time we also achieve similar results by introducing a market for contingent labor/employment lotteries.

Keywords: Endogenous firm formation, Aspiration core, Employment lotteries

1 Introduction

This paper belongs to the branch of literature initiated by Shapley and Shubik's seminal work relating economies and coalitional transferable utility (TU) games. Shapley and Shubik (1975) identify the core of any totally balanced (market) game with the set of Walrasian allocations of a corresponding pure-exchange "direct" economy. We generalize the previous result to games that are not necessarily balanced by identifying the aspiration core (Cross 1967, Bennett 1983) allocations of *any* TU-game with competitive equilibria of "direct" production economies. Furthermore, we adapt the

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coalition formation approach described in the aspiration literature to relate it with endogenous firm creation in the corresponding economy.

For games that are not balanced, the worth of the grand coalition cannot be divided among the individual players in such a way that no smaller coalition would have an incentive to form. For such games coalition formation becomes an issue as it is not clear that the grand coalition will emerge. To make the coalition formation process explicit, we define two types of direct production economies in which agents can create coalitional firms. Given a TU-game v, the members of coalition S may join efforts working in a firm whose productivity depends on $v(S)$. We label these economies as "direct" because agents are endowed with one unit of productive time which they sell to the firms in exchange for consumption. In the first type of production economy (Section 3) time is divisible and agents can spend it working simultaneously for various firms. We show that aspiration core vectors coincide with Walrasian prices of the corresponding direct production economy. Additionally, we also show that active firms in equilibrium correspond to coalitions that make such aspiration core vectors feasible (i.e., coalitions that lie in their generating collection).

However, many economic examples do not allow an agent to simultaneously be part of two different coalitions. This is why the second model (Section 4) assumes that time is indivisible. Due to the inherent non-convexity introduced, such economies do not always have a Walrasian equilibrium. We show that if agents and firms are allowed to trade lottery contracts specifying a positive probability of unemployment (as in (Rogerson 1988)), an equilibrium always exists. Equilibrium prices of these lottery production economies are in a one-to-one and onto relation with the aspiration core vectors, and firms that form with positive probability in equilibrium belong to the corresponding generating collection. Our results posit the aspiration core allocations as being the competitive market values of the individual players' participation into various coalitions.

Following Shapley and Shubik (1969) and Shapley and Shubik (1975), different types of direct economies have been associated with TU-games in the literature. Assuming perfect divisibility of labor, Inoue (2010) associates to an arbitrary TU-game v a *coalition* production economy and proves that its competitive equilibria coincide with the core of the totally balanced cover of v. Our results in Section 3 complement his findings by adding a coalition formation process based on the aspiration literature. Garratt and Qin (1997) modify Shapley and Shubik's (1975) pure-exchange economy to account for indivisibilities in agents' time and introduce trading in lotteries. They show that only super-additive games can be derived from such direct lottery economies and establish an equivalence between the core vectors of a game v and the equilibrium prices of the associated lottery economy. Sun, Trockel, and Yang (2008) analyze the role of labor indivisibilities in the context of a coalition production economy with no trade in lotteries. They show that competitive equilibria of such economy are in one-to-one and onto correspondence with the core vectors of the super-additive completion of the game (whenever the latter exist). The aspiration core of a game v coincides with the cores of v and the super-additive completion of v whenever either of the last two is not empty. Thus, our results in Section 4 generalize Garratt and Qin's (1997) and Sun, Trockel, and Yang's (2008) to arbitrary TU-games by interpreting the aspiration core vectors as the competitive market values of agents' individual skills.

The paper is organized as follows. Notation and definitions are introduced in Section 2. Section 3 assumes that agents' coalitional participations are perfectly divisible and constructs a direct production economy whose competitive equilibrium prices coincide with the aspiration core of the game. In Section 4 it is assumed that time is indivisible and agents can trade lottery contracts. We introduce the concept of a lottery equilibrium and prove the equivalence between these equilibria and the aspiration core of the game. Section 5 concludes.

2 Definitions and Notation

Let N be a finite set of n players, $\mathcal{N}$ the collection of all non-empty subsets of N, and for any $i \in N$ define $\mathcal{N}_i = \{S \in \mathcal{N} \mid S \ni i\}$. Let Δ_N (respectively $\Delta_{\mathcal{N}}$) be the unit simplex in $\mathbb{R}^N$ (respectively $\mathbb{R}^{\mathcal{N}}$), and $e_i \in \Delta_N$ (respectively $e_S \in \Delta_{\mathcal{N}}$) the vertex corresponding to $i \in N$ (respectively $S \in \mathcal{N}$). For every $S \in \mathcal{N}$, let $\mathbf{1}_S \in \{0,1\}^N$ denote the indicator function of S.

A *TU-game* (or simply a *game*) on N is a mapping $v : 2^N \to \mathbb{R}_+$ such that $v(\emptyset) = 0$. For any $S \subseteq N$, $v(S)$ is called the *worth of coalition* S. The restriction of a game v to $S \subseteq N$, is the game $v_{|S}$ on S such that $v_{|S}(T) := v(T)$ for all $T \subseteq S$. Given a game v on N, a possible outcome is represented by a *payoff vector* $x \in \mathbb{R}^N$ that assigns to every $i \in N$ a payoff x_i. Given $x \in \mathbb{R}^N$ and $S \subseteq N$, let $x(S) := \sum_{i \in S} x_i$, with the agreement that $x(\emptyset) = 0$. A payoff vector $x \in \mathbb{R}^N$ is *feasible* for coalition S if $x(S) \leq v(S)$. It is *individually feasible* if for every $i \in N$, there exists $S \in \mathcal{N}_i$ such that x is feasible for S. We say that coalition S is able to *improve upon* the outcome $x \in \mathbb{R}^N$ if $x(S) < v(S)$. A vector $x \in \mathbb{R}^N$ is *stable* if it cannot be improved upon by any coalition. The *core* of v, denoted $\mathcal{C}(v)$, is the set of stable outcomes that are feasible for N, that is,

$$\mathcal{C}(v) := \{x \in \mathbb{R}^N \mid x(S) \geq v(S)\ \forall S \subseteq N,\ x(N) = v(N)\}.$$

A stable payoff vector $x \in \mathbb{R}^N$ that is individually feasible is called an *aspiration*. We denote by $\mathcal{A}sp(v)$ the set of aspirations of game v. It is known that for any game v, $\mathcal{A}sp(v)$ is a non-empty, compact and connected set (Bennett and Zame 1988). The *generating collection* of an aspiration x is the family of coalitions S that can attain x, that is,

$$\mathcal{GC}(x) := \{S \in \mathcal{N} \mid x(S) = v(S)\}.$$

A collection of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is called *balanced* (respectively *weakly balanced*) if there exist positive (respectively non-negative) numbers $(\lambda_S)_{S\in\mathcal{B}}$ such that for every $i \in N$, $\sum_{S\in\mathcal{N}_i\cap\mathcal{B}} \lambda_S = 1$. The numbers λ_S are called *balancing weights*. A game v on N is called *balanced* if $\sum_{S\in\mathcal{B}} \lambda_S v(S) \leq v(N)$ for every balanced family $\mathcal{B}$ with balancing weights $(\lambda_S)_{S\in\mathcal{B}}$. Bondareva (1963) and Shapley (1967) showed that v is balanced if and only if $\mathcal{C}(v) \neq \emptyset$. A game v is called *totally balanced* if $v_{|S}$ is balanced for every $S \subseteq N$. For every game v, let $\bar{v}$ denote the least totally balanced set function that is greater or equal to v. The game $\bar{v}$ is called the *totally balanced cover* of v.

The *aspiration core* (Cross 1967, Bennett 1983) of a game v, denoted $\mathcal{AC}(v)$, is the set of those aspirations $x \in \mathcal{A}sp(v)$ for which $\mathcal{GC}(x)$ is weakly balanced. It is known that $\mathcal{AC}(v) = \mathcal{C}(v)$ if and only if v is balanced and $\mathcal{AC}(v) = \mathcal{C}(\bar{v}) \neq \emptyset$ for every game v.

3 Games as Production Economies with Divisible Labor

In the spirit of Shapley and Shubik (1969), we are going to associate a TU-game with every private-ownership production economy and, reciprocally, construct a private-ownership production economy (called a direct production economy) starting from any TU-game.

Let L be a finite set of goods, I a finite set of consumers and J a finite set of firms. A *private-ownership production economy* or simply an *economy* is

$$E := (L, I, J; (X_i, u_i, \omega_i)_{i\in I}, (Y^j)_{j\in J}, (\theta_i^j)_{i\in I, j\in J}),$$

where for every $i \in I$, $X_i \subseteq \mathbb{R}_+^L$, $u_i : X_i \to \mathbb{R}$ and $\omega_i \in X_i$ respectively denote agent i's consumption set, utility function and initial endowment of goods. For every $j \in J$, $Y^j \subseteq \mathbb{R}^L$ is firm j's production set and $\theta^j \in \mathbb{R}^I_+$ its distribution of shares across consumers, so that $\sum_{i \in I} \theta^j_i = 1$.[1] We restrict attention to production economies in which consumers' utility functions are quasi-linear in the same good and firms' production sets exhibit constant returns to scale.

For every non-empty subset of consumers S, let $J_S := \{j \in J \mid \sum_{i \in S} \theta^j_i = 1\}$ be the set of firms that are fully owned by consumers in S. A *consumption allocation* for E is any $x \in \prod_{i \in I} X_i$. It is called *feasible for coalition* S if there is a vector of production plans $y \in \prod_{j \in J_S} Y^j$ such that

$$\sum_{i \in S} x_i = \sum_{i \in S} \omega_i + \sum_{j \in J_S} y^j. \tag{1}$$

We denote by $\mathcal{F}(S)$ the set of feasible consumption allocations for coalition S.

Given an economy E, define the TU-game $\mathcal{V}(E)$ on I by

$$\mathcal{V}(E)(S) := \max \left\{ \sum_{i \in S} u_i(x_i) \mid x \in \mathcal{F}(S) \right\}, \tag{2}$$

for every $S \subseteq I$. A TU-game v is called a production market game if there exists a private-ownership production economy E such that $v = \mathcal{V}(E)$. Note that, as in Shapley and Shubik (1969), there are many production economies that generate the same production market game v.

Conversely, given a game v on N, we define its *direct production economy* as the private-ownership production economy

$$\mathcal{E}(v) = \left(\{L_i \mid i \in N\} \cup \{C\}, N, \mathcal{N}; (X_i, u_i, \omega_i)_{i \in N}, (Y^S)_{S \in \mathcal{N}}, (\theta^S_i)_{(i,S)} \right), \tag{3}$$

where each consumer $i \in N$ has a consumption set $X_i = \mathbb{R}^{n+1}_+$, a utility function such that $u_i(l_1, \ldots, l_n, c) = c$, and an endowment $\omega_i = (e_i, 0) \in \mathbb{R}^{n+1}_+$. Each firm $S \in \mathcal{N}$ has a production set

$$Y^S := \left\{ (l_1, ..., l_n, c) \in -\mathbb{R}^N_+ \times \mathbb{R}_+ \mid l_i = 0 \text{ if } i \notin S, c \leq \min_{i \in S} |l_i| \cdot v(S) \right\}$$

and distribution of shares $\theta^S_i = \frac{1}{|S|} \cdot \mathbf{1}_S(i)$.[2]

[1] This construction follows the description of Shapley and Shubik (1969, Section 6.1).

[2] Since firms' technologies have constant returns to scale, the distribution of shares is irrelevant for the competitive equilibrium. Our results remain true for other distributions of shares as long as for every $S \subseteq N$, consumers in S fully own firm S.

Thus, the economy $\mathcal{E}(v)$ has n consumers, $2^n - 1$ firms and $n+1$ commodities. The last commodity, denoted C, is a consumption good; the other n commodities, denoted $L_1, ..., L_n$, represent agent-specific human capital (or skilled labor). Each consumer i cares only about the amount of good C he consumes and is endowed with one unit of human capital L_i. Firms are indexed by $S \subseteq N$ and each firm S uses human capital (skilled labor) $(L_i)_{i \in S}$ to produce the consumption good.[3]

The following proposition is an analogue of Shapley and Shubik's (1969) results within our production economy framework. It shows that the only games that can be derived from a private-ownership production economy with labor divisibilities are those that are totally balanced.

Proposition 3.1 *For every game v, $\mathcal{V}(\mathcal{E}(v)) = \bar{v}$. Moreover, a game is totally balanced if and only if it is a production market game.*

Proof. Fix a game v on N. By definition, for any $S \subseteq N$, $\mathcal{V}(\mathcal{E}(v))(S) = \max \sum_{i \in S} c_i$ subject to the existence of production plans $(l^T, c^T) \in Y^T$, one for each $T \subseteq S$, satisfying

$$\sum_{i \in S} (0, c_i) = \sum_{i \in S} (e_i, 0) + \sum_{T \subseteq S} \left(l_1^T, \ldots, l_n^T, (\min_{i \in T} |l_i^T|) \cdot v(T) \right).$$

Thus, the feasibility condition for the consumption commodity is reduced to $\sum_{i \in S} c_i = \sum_{T \subseteq S} \bar{l}^T \cdot v(T)$, which implies that

$$\mathcal{V}(\mathcal{E}(v))(S) = \max \left\{ \sum_{T \subseteq S} \bar{l}^T \cdot v(T) \mid \sum_{T \in \mathcal{N}_i} \bar{l}^T = 1, \forall i \in S \right\} = \bar{v}(S),$$

as desired.

If v is a totally balanced game, then $v = \bar{v}$ and thus $\mathcal{V}(\mathcal{E}(v)) = v$, which proves that v is a production market game. Reciprocally, if $v = \mathcal{V}(E)$ for some convex economy E then its core is non-empty and thus v is balanced. For any $S \subseteq N$, the subgame $v_{|S}$ is also a production market game, as it can be generated by the restriction of E to S. As any subgame of v is balanced, we conclude that v is totally balanced. ■

We show next that competitive equilibria of a production economy are in a one-to-one and onto mapping with the aspiration core allocations of the associated TU-game.

[3]Similar constructions are used by Sun, Trockel, and Yang (2008) and Inoue (2010) to associate to every TU-game a *coalition* production economy.

Let v be a game and $\mathcal{E}(v)$ its direct production economy. A *competitive (or Walrasian) equilibrium* for $\mathcal{E}(v)$ is a vector

$$\left[(\bar{w},1) \in \mathbb{R}_+^{|N|+1}, \bar{c} \in \mathbb{R}_+^{|N|}, (-\bar{l}^S \cdot \mathbf{1}_S, \bar{l}^S v(S))_{S\in\mathcal{N}}\right]$$

of (relative) wages, allocations, and production plans such that:

1. $\pi^S = \bar{l}^S(v(S) - \bar{w}(S)) = \max_{l_S} l_S(v(S) - \bar{w}(S))$, for every $S \in \mathcal{N}$
2. $\bar{c}_i = \bar{w}_i + \sum_{S\in\mathcal{N}_i} \frac{1}{|S|}\pi^S$ for every $i \in N$
3. $\sum_{S\in\mathcal{N}_i} \bar{l}^S = 1$ for all $i \in N$
4. $\sum_{i\in N} \bar{c}_i = \sum_{S\in\mathcal{N}} \bar{l}^S v(S)$

Given a vector of relative wages $(\bar{w}_i)_{i\in N}$, each consumer i chooses an affordable consumption plan that maximizes her utility and each firm S selects an optimal production plan to maximize its profit. Given the production sets, each firm S's demand for labor must be of the form $l^S \cdot \mathbf{1}_S$, with $l^S \in \mathbb{R}_+$. Next proposition identifies the aspiration core vectors with the wages agents would receive by selling their time/skills in a competitive market and the corresponding generating collection with the set of firms that are active in equilibrium.

Proposition 3.2 *Let $\bar{x} \in \mathcal{AC}(v)$ and $(\bar{\lambda}_S)_S$ be a system of balancing weights associated with $\mathcal{GC}(\bar{x})$. Then, $[(\bar{x},1), \bar{x}, (-\bar{\lambda}_S \cdot \mathbf{1}_S, \bar{\lambda}_S v(S))_S]$ is a competitive equilibrium for $\mathcal{E}(v)$.*

Reciprocally, if $[(\bar{w},1), \bar{c}, (-\bar{l}^S \cdot \mathbf{1}_S, \bar{l}^S v(S))_{S\in\mathcal{N}}]$ is a competitive equilibrium in $\mathcal{E}(v)$, then $\bar{c} \in \mathcal{AC}(v)$ and $S \in \mathcal{GC}(\bar{c})$ whenever $\bar{l}^S > 0$.

Proof. Let $\bar{x} \in \mathcal{AC}(v)$ and $(\bar{\lambda}_S)_S$ as above. For every firm $S \notin \mathcal{GC}(\bar{x})$ it is optimal to choose $\bar{l}^S = 0$ at prices $(\bar{x},1)$ and thus remain inactive. Every firm $S \in \mathcal{GC}(\bar{x})$ is indifferent over the choice of $l^S \in \mathbb{R}_+$ at prices $(\bar{x},1)$ and, in particular, it can choose $\bar{l}^S = \bar{\lambda}_S$. Since $\mathcal{GC}(\bar{x})$ is balanced, all labor markets clear. Finally, feasibility in the consumption good holds as $\bar{x}(N) = \sum_{S\in\mathcal{GC}(\bar{x})} \bar{\lambda}_S \bar{x}(S) = \sum_{S\in\mathcal{GC}(\bar{x})} \bar{\lambda}_S v(S) = \sum_{S\in\mathcal{N}} \bar{\lambda}_S v(S)$. Thus, the vector $\left[(\bar{x},1), \bar{x}, (-\bar{\lambda}_S \cdot \mathbf{1}_S, \bar{\lambda}_S v(S))_S\right]$ is a competitive equilibrium for $\mathcal{E}(v)$.

Let now $\left[(\bar{w},1), \bar{c}, (-\bar{l}^S \cdot \mathbf{1}_S, \bar{l}^S v(S))_{S\in\mathcal{N}}\right]$ be a competitive equilibrium in $\mathcal{E}(v)$. Since production sets exhibit constant returns to scale, profits equal zero for all $S \in \mathcal{N}$. Then $\bar{c}_i = \bar{w}_i$ for every $i \in N$ and $v(S) \leq \bar{w}(S) = \bar{c}(S)$ for every $S \in \mathcal{N}$. Therefore $\bar{c}$ is stable. Moreover, $\bar{l}^S = 0$ for every S for which $v(S) < \bar{w}(S)$. The market clearing condition implies then that $\mathcal{GC}(\bar{c})$

is balanced, with balancing weights $(\bar{l}^S)_S$, and thus $\bar{c} \in \mathcal{AC}(v)$. ■

Our treatment extends the analogies between games and direct economies presented in Shapley and Shubik (1975) and Inoue (2010). In particular, we show that not only payoffs, but formed coalitions (in game v) and productive firms (in economy $\mathcal{E}(v)$), coincide. For any coalition $S \in \mathcal{N}$, the balancing weight λ_S is equal to the amount of time l_S firm S is active in the Walrasian equilibrium of the corresponding direct production economy.

3.1 An Example

Consider the following game: $v(\emptyset) = v(1) = v(2) = v(3) = 0, v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1$. This game has an empty core, but its aspiration core is $\mathcal{AC}(v) = \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The generating collection of its unique element is $\mathcal{GC}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \{\{1,2\}, \{2,3\}, \{1,3\}\}$. We are going to illustrate next how the aspiration core allocation can be obtained via a decentralized market mechanism.

Consider an economy with three agents (call them truck drivers) and three firms.[4] Agents have identical skills and each is endowed with one unit of time which can be supplied as labor. Firms have identical technologies (each owns a truck) and hire labor to produce the same output good (deliveries). The production set of a multi-agent firm S is $Y^S = \left\{(l_1, l_2, l_3, c) \in -\mathbb{R}^3_+ \times \mathbb{R}_+ \mid l_i = 0 \text{ if } i \notin S, c \leq \min_{i \in S} |l_i|\right\}$. Thus every delivery needs the labor input of two truck agents. Truck drivers only care about the number of deliveries they make and have no disutility of labor.

Assume first that both labor and the output good are divisible and thus agents can choose to work part-time for various firms. This case serves as an illustration of Proposition 3.2. It is easy to check that $w = \frac{1}{2}$ is an equilibrium wage at which each agent works half-time for two firms and each firm employs exactly 2 workers.

If labor is indivisible (agents cannot receive part-time contracts) then an equilibrium does not exist. At a wage $w = \frac{1}{2}$, each agent chooses to work for exactly one firm. On the other hand, no firm wants to hire only one or all three workers, because its profits would be negative. Consequently there is no allocation of workers to firms and no equilibrium. Non-existence of a competitive equilibrium in this case is directly related to the emptiness of the core of the game v.

[4]Three firms are enough to illustrate the equivalence between competitive wages and aspiration core allocations in this example because the generating collection of the aspiration core allocation has three elements.

Assume next that labor is still indivisible, but agents can trade employment lotteries. For example, workers can submit job applications to more than one firm and randomize over which offer to accept. Firms can also offer employment contracts that stipulate a probability of being laid off (or a probability of delivery cancellation). In this case $w = \frac{1}{2}$ is again an equilibrium wage. At this wage, each driver chooses to submit exactly two job applications and accept each firm's offer with equal probability. Each firm chooses to hire exactly two drivers, offering them employment contracts that carry a 50% chance of delivery cancellation. There are three essentially different outcomes arising from these equilibrium wages. Each of them occurs with a probablity of $\frac{1}{3}$:

1. Agents 1 and 2 work for one firm and the other two firms are inactive,
2. Agents 2 and 3 work for one firm and the other two firms are inactive,
3. Agents 1 and 3 work for one firm and the other two firms are inactive.

The following section formalizes these results, making them applicable to arbitrary TU-games.

4 Indivisibilities, lotteries and the aspiration core

We modify the analysis of the previous section by assuming that goods $L_1, ...L_n$ are indivisible, while the consumption good C remains perfectly divisible. Given a game v on N, we define its *indivisible direct production market* as

$$\mathcal{E}^{indiv}(v) = \left[\{L_i \mid i \in N\} \cup \{C\}, N, \mathcal{N}, (X_i, u_i, \omega_i)_{i \in N}, (Y^S)_{S \in \mathcal{N}}, (\theta_i^S)_{(i,S)}\right],$$

where $X_i = \{0,1\}^n \times \mathbb{R}_+$, $u_i(l_1, \ldots, l_n, c) = c$, and $\omega_i = (e_i, 0)$ for each consumer $i \in N$. Each firm S has a production set $Y^S := \{k \cdot (-\mathbf{1}_S, v(S)) \mid k \in \mathbb{N}\}$ and shares $\theta_i^S = \frac{1}{|S|} \cdot \mathbf{1}_S(i)$.

Note that if agents only work for the grand coalition time divisibility becomes irrelevant. Thus, Proposition 3.2 implies $x \in \mathcal{C}(v)$ if and only if x is a competitive allocation for $\mathcal{E}^{indiv}(v)$. Moreover, as proved by Sun, Trockel, and Yang (2008), the indivisible direct production market does not have an equilibrium unless the *super-additive completion* of v, is balanced.[5] We show next that, if firms and consumers are allowed to sign employment contracts

[5]The *super-additive completion* of v is defined as a game $\tilde{v}$ such that $\tilde{v}(S) = v(S)$ if $S \neq N$ and $\tilde{v}(N) = \max_{\mathcal{B} \in \mathcal{P}} \sum_{S \in \mathcal{B}} v(S)$, where $\mathcal{P}$ denotes the set of all partitions of N.

contingent on the outcome of a lottery (see Rogerson (1988) for a related model), equilibrium wages always exist and coincide with the aspiration core vectors of the corresponding game v.

Assume that (skilled) labor is indivisible, but consumers and firms may choose to default on their labor contracts. Consumers may contemplate switching between equally-paying jobs, while firms can layoff workers and get out of business. However, rather than modeling strictly enforceable contracts and punishments for default, we design our model such that the probabilities of default will be embedded in the equilibrium market prices. We assume therefore that consumers and firms trade in *labor* or *employment lotteries* specifying, for each party, a probability of employment termination as described below.

A *labor lottery* for agent i is a vector $p_i \in \Delta_{\mathcal{N}_i}$ such that $\sum_{S\in\mathcal{N}_i} p_i^S = 1$. Thus, p_i^S specifies the probability with which agent i chooses to work for firm S (or, alternatively, $1 - p_i^S$ can be interpreted as the probability that i will terminate his/her contract with firms S, if hired). Given a wage level w_i, agent i chooses a probability distribution over the firms $S \in \mathcal{N}_i$. We assume that consumers are risk-neutral and thus the utility consumer i derives from choosing the labor lottery p^i and consumption c is:

$$U_i(p_i, c) := \sum_{S\in\mathcal{N}_i} p_i^S u_i(e^S, c) = c$$

for all $(p_i, c) \in \Delta_{\mathcal{N}_i} \times \mathbb{R}_+$.

An *employment lottery* for firm S specifies a probability, $\phi_S \in [0,1]$ of maintaining employment from that firm or, equivalently, a probability $1-\phi_S$ of being laid off. Alternatively, one can interpret ϕ_S as the probability that S remains in business. Each firm S chooses an employment lottery and, contingent on being active, an operating level (labor force size) $k_S \in \mathbb{N}$. Each firm S is assumed to maximize its expected profits and thus it solves

$$\max \left\{ \phi_S \cdot k_S \left(v(S) - w(S) \right) \mid \phi_S \in [0,1], k_S \in \mathbb{N} \right\}. \tag{4}$$

As opposed to the standard employment lottery models which assume a continuum of agents (e.g. Rogerson (1988)), our economy is finite and thus we cannot rely on the law of large numbers to ensure labor market clearing. Along the lines of Garratt (1995) we say that a set of labor/employment lotteries is feasible if its elements are the marginals of some (auctioneer-run) joint lottery on the set of feasible labor contracts. More precisely, we define a *labor contract* as a vector $x \in \{0,1\}^{\mathcal{N}}$, in which the component x_S is equal to 1 if and only if firm S is active (and thus every consumer $i \in S$ is employed full-time). A labor contract is *feasible* if $[x_S = x_{S'} = 1] \Rightarrow [S \cap S' = \emptyset]$ for all $S \neq S'$.

Note that feasibility of labor contracts only requires that there is no excess demand for labor/human capital. It does *not* require that the labor market clears. For every feasible labor contract x, define $T(x) := \bigcup \{S \mid x_S = 1\}$ as the set of employed agents. At a feasible labor contract, $T(x)$ may be a strict subset of N. Denote the set of all feasible labor contracts by $\mathcal{X}$, and consider an arbitrary probability distribution γ on $\mathcal{X}$. Then the probability that firm S is active is $\sum_{\{x|x_S=1\}} \gamma(x)$, while the probability that consumer i is employed is $\gamma_i := \sum_{\{x|T(x)\in\mathcal{N}_i\}} \gamma(x)$.

Definition 4.1 *A set of labor and employment lotteries $((p_i)_{i\in N}, (\phi_S)_{S\in\mathcal{N}})$ is feasible if*

1. *There exists $\gamma \in \Delta_{\mathcal{X}}$ such that $\phi_S = \sum_{\{x|x_S=1\}} \gamma(x)$, for all $S \in \mathcal{N}$,*
2. *$p_i^S = \frac{\phi_S}{\sum_{T\in\mathcal{N}_i} \phi_T}$, for every $S \subseteq N$ and every $i \in S$, and $p_i^S = p_j^S$ if $i, j \in S$.*

The first condition is a compatibility condition for labor demand. It requires that the probability of firm S operating coincides with the marginal of a joint probability distribution over the set of feasible labor contracts. The second condition requires that the probability that agent i assigns to working for firm S is exactly the probability of firm S operating, conditional on i being employed. The second part of condition 2 captures the labor complementarities embedded in firms' technologies. Note that the two conditions imply that $\sum_{S\in\mathcal{N}_i} \phi_S = \gamma_i > 0$ and $\gamma_i = \gamma_j$ for all $i, j \in N$.

Definition 4.2 *An equilibrium for the direct lottery market is a list of vectors*

$$\left[(\bar{w}_i)_i, (\bar{p}_i)_i, (\bar{\phi}_S)_S, (\bar{k}_S)_S\right]$$

such that

1. *$\bar{p}_i \in \Delta_{\mathcal{N}_i}$ for every $i \in N$*
2. *$(\bar{\phi}_S, \bar{k}_S)$ solves (4) for every $S \subseteq N$*
3. *$k_S = 1$, for all $S \subseteq N$*
4. *$((\bar{p}_i)_i, (\bar{\phi}_S)_S)$ is feasible according to Definition 4.1.*

We can now relate aspiration core allocations with equilibrium wages.

Theorem 4.3 *If $\bar{w} \in \mathcal{AC}(v)$ and $(\lambda_S)_S$ is a system of balancing weights associated with $\mathcal{GC}(\bar{w})$, then*

$$\left[(\bar{w}_i)_{i\in N}, ((\lambda_S)_{S\in\mathcal{N}_i})_{i\in N}, \left(\frac{\lambda_S}{\Lambda}\right)_{S\in\mathcal{N}}, (k_S = 1)_{S\in\mathcal{N}}\right]$$

is a competitive equilibrium for the direct lottery market, where $\Lambda := \sum_{S\in\mathcal{N}} \lambda_S$.

Reciprocally, if $[(\bar{w}_i)_{i\in N}, (\bar{p}_i)_{i\in N}, (\bar{\phi}_S)_{S\subseteq N}, (\bar{k}_S)_{S\subseteq N}]$ is a competitive equilibrium for the direct lottery market, then $\bar{w} \in \mathcal{AC}(v)$ and $S \in \mathcal{GC}(\bar{w})$ whenever $\bar{\phi}_S > 0$.

Proof. Let $\bar{w} \in \mathcal{AC}(v)$ and $(\lambda_S)_S$ a system of balancing weights associated with $\mathcal{GC}(\bar{w})$. Define $\bar{p}_i^S := \lambda_S$, $\bar{\phi}_S := \frac{\lambda_S}{\Lambda}$ and $\gamma \in \Delta_{\mathcal{X}}$ such that $\gamma(x) = \bar{\phi}_S$ if $x = e_S$ for some $S \in \mathcal{N}$ and $\gamma(x) = 0$ otherwise. Then $((\bar{p}_i)_{i\in N}, (\bar{\phi}_S)_{S\in\mathcal{N}})$ is feasible, being supported by the joint lottery $\gamma \in \Delta_{\mathcal{X}}$. Moreover, since $\bar{w} \in \mathcal{AC}(v)$, $\bar{w}(S) \geq v(S)$ and thus $(\bar{\phi}_S, 1)$ is an optimal choice for firm S, which generates an expected profit of 0.

Reciprocally, if $[(\bar{w}_i)_{i\in N}, (\bar{p}_i)_{i\in N}, (\bar{\phi}_S)_{S\subseteq N}, (\bar{k}_S)_{S\subseteq N}]$ is an equilibrium for the direct lottery market, then $\bar{w}(S) \geq v(S)$, otherwise firm S would make infinite profits. Profit maximization also dictates that $\bar{\phi}_S > 0$ only if $\bar{w}(S) = v(S)$. On the other hand, feasibility implies that $\sum_{S\in\mathcal{N}_i} \bar{\phi}_S > 0$ and thus, for every $i \in N$ there exists $S \in \mathcal{N}_i$ such that $\bar{\phi}_S > 0$ and $\bar{w}(S) = v(S)$, which implies that $\bar{w}$ is an aspiration. In addition, $\lambda_S := \frac{\phi_S}{\sum_{T\in\mathcal{N}_i} l_T}$ does not depend on i and $\sum_{S\in\mathcal{N}_i} \lambda_S = 1$ for every $i \in N$. This proves that $\mathcal{GC}(\bar{w})$ is balanced and thus $\bar{w} \in \mathcal{AC}(v)$. ■

An immediate consequence of Theorem 4.3 is that the core of a game v is non-empty if and only if the direct lottery market has a degenerate equilibrium in which $p_i^N = 1$ for every $i \in N$, $l_N = 1$ and $\phi_S = 0$ for every $S \subsetneq N$. Thus, all consumers are employed by one firm and there is no default in the labor-employment contracts. Each agent receives a wage (and utility) equal to his/her payoff at a core allocation. This is equivalent to saying that the grand coalition forms and its worth is split among agents according to some core vector.

On the other hand, the game v has a non-empty c-core if and only if the direct lottery market has an equilibrium for which $\gamma_i = 1$ for all $i \in N$. In this case it is still true that each agent is employed but, unless the game is balanced, several firms may be active. Wages received are elements in the c-core of the game. Thus, our results generalize those of Sun, Trockel, and Yang (2008).

If the c-core of the TU-game is empty, then each player faces a positive probability of being unemployed and thus, in every realization of the joint lottery, the labor market is in excess supply. Firms that are active at a particular realization of the equilibrium lottery correspond to elements of the generating collection and wages paid are elements of the aspiration core of the game.

5 Final Remarks

Most cooperative solution concepts do not simultaneously address the allocation and coalition formation problems. For example the core, if non-empty, exogenously dictates the formation of the grand coalition. Zhou (1994) moves a step forward by defining a new type of bargaining set which addresses both questions. On the down side, Zhou's (1994) bargaining set cannot be decentralized using a market economy (Anderson, Trockel, and Zhou 1997). The aspiration core endogenously proposes both payoffs and coalitions and this paper endows it with the link to competitive equilibrium that Zhou's (1994) bargaining set is lacking. Additionally, the aspiration core not only includes, but it coincides with the core when the latter is non-empty.

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